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An Exponential Linear Model Matching for a Closed Kinematics Chain.

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Abstract

In this paper we propose an implicit linear control law for a two degree freedom manipulator whose aim is to stabilize and match a linear model. We show that for any finite initial condition there exists a sufficient small control parameter, ε, such that the model matching is exponentially achieved.

Keywords: Parallel Robots, Lyapunov 2nd method, Stability, Implicit Systems, PD control law.

1 Introduction

Control of the closed kinematic chain mechanism "*Parallel Robots*" based on the Lyapunov approach, has been the subject of sustained interest for, both, mechanical and robotics engineering. This area is considered a challenging field for potential applications in future developments for many industries. For example, we could mention the manufacturing car and aerospace industries and applications such as fast assembly lines, flight simulators, robotic matchines (see : [Fither,1986], [Nguyet et al 1993], [Merlet,1990] and [Dunlop and Garcia, 2002]). The main feature of these manipulators is that the links are connected in series as well as in parallel combinations, forming one or more closed-link loops. Besides, not all the joints of parallel robots are actuated (see [Merlet, 2000] and [Dunlop and Jones, 1998]). The advantages of using *Parallel Robots* should be stressed, since, they present high precision positioning capabilities and non cumulative link errors due to their high structural rigidity. Additionally, they have higher strength-to-weight ratios in comparison with conventional series manipulators (see:[Lebret and Lewis 1993] and [Nguyen, Pooran, and Premack, 1998]). However, their dynamics analysis presents an extremely difficult theoretical problem. For its mathematical models are very complex to analyze and to manage. For instance, we mention the well-known Stewart Platform-based manipulator with 6 freedom degrees whose mathematical model includes several nonlinear differential ordinary equations and many internal algebraic restrictions [Merlet, 1990 and 2000]. To overcome this overwhelming complexity, it is common to despise the nonlinear dynamics (Coriolis acceleration or dynamical interactions between loops) and the restrictions are not considered (see see [Lebret and Lewis 1993] , [Nguyen, Pooran, and Premack, 1998] and [Dunlop and Jones, 1998]). Consequently, the control laws design have been barely developed.

In this paper we propose a simple linear implicit control law for the most elemental but fundamental closed kinematic chain (**CKC**), namely a "*triangle chain*" (see Figure 1). The main feature of the presented closed loop system is that it allows to stabilize and to match a linear model in an exponential fashion. The used implicit control law is based on the previous results presented in ([Aguilar and Bonilla, 1998 and 1999, and Bonilla and Malabre, 1992). These authors synthesized pure derivators by means of the desing of implicit control laws. It is worth mentioning that the fundamental **CKC** permits to understand and to model more complex manipulators such as the Stewart Platform.

The rest of this paper is organized as follows ; Section 2 is dedicated to the study and overview of some mechanical properties of the **CKC** . In Section 3 we present the control law. Section 4 is devoted to the study of asymptotic stability and exponential stability of the close loop system. Section 5 analyses the exponential model matching. In Section 6 we give some conclusions about Theorems 1 and 3. Finally, in Section 7 we present the Appendix.

Let us finish this section setting the following notation: $\lambda_M\{X\}$ and $\lambda_m\{X\}$ stands for the maximum and minimum eigenvalues of the symmetric matrix X, and

$$C_i = \cos\theta_i, S_i = \sin\theta_i, i = 1, 2.$$

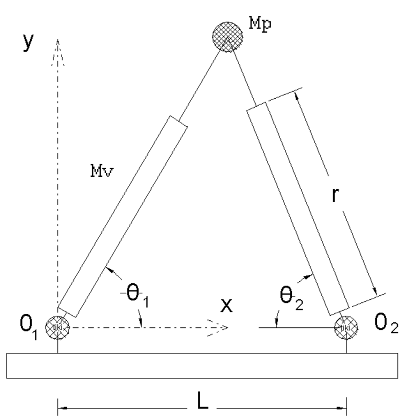

Figure 1, the **CKC.**

2 Mechanical Properties Of The CKC

This section studies basic mechanical properties of the fundamental **CKC**. Also, we introduce the kinematic chain and a dynamical model. Finally, some important properties of the **CKC** are introduced.

2.1 Fundamental CKC

Let us consider the following basic triangle chain :

It is basically constituted by two electrical pistons linked to each other at one extremity by a ball-and-socket joint; the other extremities of the pistons are linked to a fixed beam which is the base of the platform, each end of the actuator link being mounted on the platform base by a rotatory joint whose axes are parallel to each other. There is a mechanical load (a mass M_p) at the ball-and-socket joint. The left piston can move around the fixed point 0_1 and the right can move around the fixed point 0_2, and L is the fixed separation between 0_1 and 0_2 (length of the base platform). The origin of coordinates is chosen at the left piston beam joint 0_1 , the $x-$ axis lies at the base of the platform and the $y-$ axis points upward to the base. To define the Cartesian variables we proceed to assign two independent coordinates $x(t)$ and $y(t)$. The f_i are the forces supplied by pistons, r the length of the main body of the piston having a mass M_v (we have assumed that the mass is concentrated at $r/2$ and we have neglected the piston rod mass), J is the inertia moment of each actuator, and $l_i(t)$ are the variable lengths of the pistons satisfying $l_i(t) \geq r$.

Let us express the angles θ_i and the lengths l_i in Cartesian coordinates:

$$\begin{aligned}\theta_1 &= \arctan\left(\frac{y}{x}\right); \theta_2 = \arctan\left(\frac{y}{L-x}\right);\\ l_1 &= \sqrt{x^2+y^2}; l_2 = \sqrt{(L-x)^2+y^2}.\end{aligned}$$

2.2 Dynamical Model

The following model has been obtained in [Bonilla and Salazar]:

$$M(q)\ddot{q} + \Phi(q,\dot{q})\dot{q} + G(q) = f_q \tag{1}$$

where q is the coordinate vector $q = [x\ y]^T$, f_q is the force vector $f_q = [f_x\ f_y]^T$, with

$$f_x = f_1C_1 - f_2C_2 \quad ; \quad f_y = f_1S_1 + f_2S_2;$$

$M(q)$ is the symmetrical inertia matrix

$$M(q) = \begin{bmatrix} M_p + J(\frac{S_1^2}{l_1^2} + \frac{S_2^2}{l_2^2}) & J(\frac{C_2S_2}{l_2^2} - \frac{C_1S_1}{l_1^2}) \\ J(\frac{C_2S_2}{l_2^2} - \frac{C_1S_1}{l_1^2}) & M_p + J(\frac{C_1^2}{l_1^2} + \frac{C_2^2}{l_2^2}) \end{bmatrix},$$

and $\Phi(q,\dot{q})$ is the Coriolis matrix

$$\Phi(q,\dot{q}) = J\begin{bmatrix} \frac{\delta_2S_2^2}{l_2^2} + \frac{\delta_1S_1^2}{l_1^2} & \frac{\delta_2C_2S_2}{l_2^2} - \frac{\delta_1C_1S_1}{l_1^2} \\ \frac{\delta_2C_2S_2}{l_2^2} - \frac{\delta_1C_1S_1}{l_1^2} & \frac{\delta_2C_2^2}{l_2^2} + \frac{\delta_1C_1^2}{l_1^2} \end{bmatrix},$$

δ_1 and δ_2 are auxiliary variables

$$\delta_1 = \frac{\dot{l}_1}{l_1} \quad ; \quad \delta_2 = \frac{\dot{l}_2}{l_2} \ ,$$

and $G(q)$ is the gravity vector:

$$G(q) = \begin{bmatrix} \frac{M_v gr}{2}\left(-\frac{C_1 S_1}{r_1} + \frac{C_2 S_2}{r_2}\right) \\ M_p g + \frac{M_v gr}{2}\left(\frac{C_1^2}{r_1} + \frac{C_2{}^2}{r_2}\right) \end{bmatrix}.$$

2.3 Dynamical Properties

We mention the following properties:

P.1) $|\delta_i| \le \frac{\|\dot{q}\|}{r}$ and $\dot{\theta}_i \le \frac{\|\dot{q}\|}{r}$.

P.2) Inertia matrix is upper and lower bounded as: $0 < \underline{\mu} \le \lambda_m\{M(q(t))\} \le \|M(q(t))\| \le \bar{\mu}$; where $\underline{\mu} = \frac{M_p^2}{\bar{\mu}}$ and $\bar{\mu} = 2M_p + \frac{4J}{r^2}$.

P.3) Corilis matrix is bounded, *i.e.*, $\left\|\Phi(q(t),\dot{q}(t))\right\| \le K_\Phi \left\|\dot{q}(t)\right\|$, were $K_\Phi = \frac{2\sqrt{2}J}{r^3}$.

P.4) $N = \dot{M}(q) - 2\Phi(q,\dot{q})$ is a skew symmetric matrix and hence the expression $q^T N q$ is identically zero for all $q \in R^2$.

P.5) $\|G(z) - G(w)\| \le K_G \|z - w\|$, were $K_G = \frac{6M_v g}{r}$.

P.6) Let E be a matrix such that $E(q) = \mu M^{-1}(q) - I$ where $\mu = a\bar{\mu}$ for $1 < a$ then $\lambda_m\{E(q)\} = a - 1 > 0$.

The reader is referred to revise [Aguilar and Bonilla, 1999] to check the previous properties.

2.4 Energy Equations

Now, we introduce the potential and kinetic energies, which permit us state a Lyapunov function.

Let V_1 be the kinetic energy created by the translational motion of load mass M_p, plus the kinetic energy produced by the rotational motion of each actuator:

$$V_1(\dot{q}) = \frac{\dot{q}^T M(q)\dot{q}}{2}.$$

Let the potential energy E_p of the load mass plus the potential energy stored in each actuator:

$$E_p(q) = M_p g y + \frac{1}{2} M_v gr S_1 + \frac{1}{2} M_v gr S_2.$$

3 Control Law

Consider the following linear implicit control law:

$$f_q = \frac{\mu}{\varepsilon}\begin{bmatrix} \chi_1 \\ \nu_1 \end{bmatrix} - \mu k_0 q - \mu k_{1\varepsilon}\dot{q} + \mu k_{0\varepsilon} R_q + G(R_q), \quad (2)$$

where

$$\begin{bmatrix} \dot{\chi}_1 \\ \dot{\chi}_2 \end{bmatrix} = A \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix} + B \begin{bmatrix} x \\ \dot{x} \end{bmatrix},$$

$$\begin{bmatrix} \dot{\nu}_1 \\ \dot{\nu}_2 \end{bmatrix} = A \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} + B \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$$

and

$$A = \begin{bmatrix} -k_1 & 1 \\ -k_0 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} -k_0 & -\beta \\ -k_0\beta & (k_0 - k_1\beta) \end{bmatrix},$$

here R_q is the constant reference vector defined as $R_q = \begin{bmatrix} r_x & r_y \end{bmatrix}^T$ and

$$k_{0\varepsilon} = \frac{\beta + \varepsilon k_0}{\varepsilon} \quad ; \quad k_{1\varepsilon} = \frac{1 + \varepsilon k_1}{\varepsilon} \ , \quad (3)$$

where μ , ε and β are three positive constants and k_0 and k_1 are the positive coefficients of the Hurwitz polynomial $\lambda^2 + k_1\lambda + k_0$.

Notice that this control law can also be expressed as follows:

$$f_q = -\mu k_{0\varepsilon}(q - R_q) - \mu k_{1\varepsilon}\dot{q} + \frac{\mu}{\varepsilon}\Omega + G(R_q) \quad (4)$$

where $\Omega = [\chi_1 + \beta(x - r_x) \ \nu_1 + \beta(y - r_y)]^T$ is the solution of the following vectorial ordinary differential equation:

$$\ddot{\Omega} + k_1\dot{\Omega} + k_0\Omega = 0$$

Clearly, the above differential equation can be written as[1]:

$$\dot{w}_\Omega = A_\Omega w_\Omega \quad (5)$$

where

$$w_\Omega = \begin{bmatrix} \Omega \\ \dot{\Omega} \end{bmatrix}, \quad \text{where} \quad A_\Omega = \begin{bmatrix} 0_2 & I_2 \\ -k_0 I_2 & -k_1 I_2 \end{bmatrix},$$

Remark 1: Due to the fact that k_0 and $k_1$are positives, we can guarantee that A_Ω is Hurwitz, therefore, there exist a positive definite matrix P such that the following Lyapunov equation is fulfilled $A_\Omega^T P + P A_\Omega = -I_4$.

Remark 2: It can be easily seen that

$$\left\|e^{A(t-\tau)}\right\| \le c_0 e^{-\lambda_k(t-\tau)}; \|w_\Omega(t)\| \le c_0\omega_0 e^{-\lambda_k t}; \quad \omega_0 = \|w_\Omega(0)\|, \quad (6)$$

holds.

4 Asymptotic Stability

In this section we study the stability of system (1) when it is feedback by the implicit control law (2). Using Lyapunov 's second method we show that if the positive coefficient ε of the I.C.L. is chosen less than a specific bound, we can guarantee that the closed loop system is asymptotically stable.

[1] I_n and O_n stand for the identity and the zero square matrices with n rows.

Substituting the control law (4) into system (1) we get the following closed-loop description,

$$M(q)\ddot{q} + \Phi(q,\dot{q})\dot{q} + G(q) = \frac{\mu\Omega}{\varepsilon} + G(R_q) - \mu k_{0\varepsilon}(q - R_q) - \mu k_{1\varepsilon}\dot{q}. \quad (7)$$

Now, we define the state ϖ as:

$$\varpi = \left[\begin{array}{ccc} (q - R_q)^T & \dot{q}^T & w_\Omega^T \end{array} \right]^T.$$

Theorem 1 *The closed loop system (7) is asymptotically stable ($\mathcal{AS}$) if*

$$\frac{1}{\varepsilon} > \max\left\{ \frac{K_G - \mu k_0}{\beta\mu}, \frac{1-\mu k_0}{\beta\mu} \right\}. \quad (8)$$

Proof:

First, the following function is proposed

$$V_A(\varpi) = V_G(q) + \beta_0 + V_1(\dot{q}) + V_2(q) + V_3(w_\Omega) \quad (9)$$

where V_G is the potential gravity energy,

$$V_G(q) = E_p(q) - E_p(R_q) - G^T(R_q)(q - q(0)).$$

and

$$V_1(\dot{q}) = \frac{\dot{q}^T M(q)\dot{q}}{2}; \quad V_2(q) = \frac{\mu k_{0\varepsilon}(q-R_q)^T(q-R_q)}{2}; \quad V_3(w_\Omega) = \frac{\mu}{2\varepsilon} w_\Omega^T P w_\Omega^T.$$

Clearly, $V_G(q)$ satisfies the following property (see [Aguilar and Bonilla, 1999] fore more detail)

$$-\beta_0 - \frac{\|q - R_q\|^2}{2} \leq V_G(q)$$

where $\beta_0 = 2M_v gr + K_g \|R_q - q(0)\| + \frac{(M_p g + K_g)^2}{2}$. By another hand, if we select $\mu k_{0\varepsilon} > 1$ then $V_A(\varpi) > 0$ for any $\varpi \neq 0$ (also, see relation 33 of [Aguilar and Bonilla, 1999]).

Next we compute the derivative of V_A. From (7) and **P.4**, we obtain after some algebraic manipulations the following relation (recall definition of w_Ω)

$$\dot{V}_A(\varpi) = -\mu k_{1\varepsilon} \|\dot{q}\|^2 + \frac{\mu}{\varepsilon}\dot{q}^T\Omega - \frac{\mu}{\varepsilon}\|w_\Omega\|^2.$$

Thus, the above relation can be bounded as

$$\dot{V}_A(\varpi) \leq -(\mu k_{1\varepsilon} - \frac{\mu}{2\varepsilon})\|\dot{q}\|^2 - \frac{\mu\|\Omega\|^2}{2\varepsilon} - \frac{\mu}{\varepsilon}\left\|\dot{\Omega}\right\|^2.$$

From (3) it follows that $k_{1\varepsilon} > \frac{1}{2\varepsilon}$, hence, we have that $\dot{V}_A(\varpi) \leq 0$.

Since V_A is positive definite and $\dot{V}_A$ is only negative semidefinite, we have only proved stability in the sense of Lyapunov, namely, that the error and velocities are bounded. To complete the proof, we invoke, next, LaSalle's Theorem.

Define the set:

$$S = \left\{ \varpi \middle| \dot{V}_A(\varpi) = 0 \right\} = \left\{ \varpi \,|\, \varpi = \left[\begin{array}{ccc} c^T & 0 & 0 \end{array} \right] \right\},$$

with $dc/dt = 0$. Now, taking any trayectory $\varpi(t)$ belonging to S, we have

$$\mu k_{0\varepsilon} \|(c - R_q)\| = \|G(c) - G(R_q)\| \leq K_G \|(c - R_q)\|,$$

evidently, if $\mu k_{0\varepsilon} > K_G$ we must have $c = R_q$. Therefore, we conclude that the largest invariant set contained inside the set S is constituted by the point, $\varpi_e = (q^T = R_q^T, \dot{q} = 0, w_\Omega^T = 0)$. According to the LaSalle's Theorem all trajectories of the closed loop system asymptotically converge towards the invariant set contained in S, which is constituted by the equilibrium point $\varpi_e = 0$.

5 Exponential Model Matching

The objective in exponential model matching is to design an input control law for a given system so that the response (behavior) of the closed-loop system asymptotically exponentially matches that of a prescribed, driven model and so that the closed-loop system is internally stable [2] (see: [Di Benedetto and Grizzlle, 1994], [Huijberts and Nijmeijer, 1990], [Byrnes *et. al.* 1998], [Di Benedetto and Isidori, 1986]. Another form to see it is to render the closed-loop response of the **CKC** provided that follows a exponentially stable invariant linear system (**ESILS**). Evidently, if a non linear system behaves as an **ESIL** then we can claim the closed-loop system is robust to external perturbations and non modeled dynamics. However, to achieve this it is required to design a high gains controller causing the actuator to do a big effort.

Thus, considering again the closed loop system defined by equation (7), and, we show that the closed-loop response of the **CKC** follows a **ESILS**, when the parameter ε is chosen less than a specific bound.
To carried on this, we proceed as follows:

1) We first present an error state space realization of the closed loop system. 2) We next study some useful properties of the error equation to be used in a Lyapunov equation. 3) We finally present in Theorem 3, a sufficient condition which guarantees the exponential model matching.

Before proceeding, consider a dynamical trajectory ζ solution of the differential equation:

$$\dot{\zeta}(t) + \beta(\zeta(t) - R_q) = \Omega(t) \quad (10)$$

[2] In this case we desired to follow a Hurwitz invariant linear system.

Notice that the above equation satisfies the following inequalities:

$$\begin{array}{c}\|\zeta(t)-R_q\| \leq \alpha_0 e^{-\lambda^* t}, \left\|\dot{\zeta}(t)\right\| \leq \alpha_1 e^{-\lambda^* t}, \\ \left\|\ddot{\zeta}(t)\right\| \leq \alpha_2 e^{-\lambda^* t},\end{array} \tag{11}$$

were (recall (6)):

$$\begin{array}{l}\lambda^* = \min\{\beta, \lambda_k\}; \\ \alpha_0 = \sqrt{\|\zeta(0)-R_q\|^2 + (\frac{c_0\omega_0}{\beta-\lambda_k})^2}; \\ \alpha_1 = \sqrt{2(c_0\omega_0)^2 + 2(\beta\alpha_0)^2}; \\ \alpha_2 = \sqrt{2(c_0\omega_0)^2 + 2(\beta\alpha_1)^2}.\end{array} \tag{12}$$

It should be mention that if all the initials conditions are selected such that $\zeta(0) - R_q = \Omega(0) = 0$ then the constants α_0, α_1 and α_2 are equal to cero.

5.1 Error State Space Realization

The closed-loop behavior (7) can be also expressed as follows

$$M(q)\ddot{q} + \Theta = -\mu k_{1\varepsilon}\dot{q} - \mu k_{0\varepsilon}(q - R_q) + \frac{\mu\Omega}{\varepsilon}, \tag{13}$$

where $\Theta = \Phi(q,\dot{q})\dot{q} + G(q) - G(R_q)$.

Defining the tracking error as

$$\mathbf{e}^T(t) = \left[(q-\zeta)^T \ , (\dot{q}-\dot{\zeta})^T\right]^T,$$

substituting the tracking error into equation (13) and using **P.6** , we have

$$\dot{\mathbf{e}}(t) = A_\varepsilon \mathbf{e}(t) + \begin{bmatrix} 0 \\ -E(q)K_\varepsilon \mathbf{e}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \Delta(t) \end{bmatrix}, \tag{14}$$

where $K_\varepsilon = [k_{0\varepsilon}I_2 \ k_{1\varepsilon}I_2]$ and

$$\Delta(t) = -M^{-1}\Theta - \ddot{\zeta} - M^{-1}(k_0(\zeta - R_q) + k_1\dot{\zeta})$$

$$A_\varepsilon = \begin{bmatrix} O_2 & I_2 \\ -k_{0\varepsilon}I_2 & -k_{1\varepsilon}I_2 \end{bmatrix}.$$

Remark 3: Since $k_{0\varepsilon}$ and $k_{1\varepsilon}$ are positives we can guarantee that A_ε is a Hurwitz matrix. Hence, there exist P_ε and Q_ε,

$$P_\varepsilon = \begin{bmatrix} 2k_{0\varepsilon}k_{1\varepsilon}I & k_{0\varepsilon}I \\ k_{0\varepsilon}I & k_{1\varepsilon}I \end{bmatrix}; \quad Q_\varepsilon = \begin{bmatrix} 2k_{0\varepsilon}^2 I & 0 \\ 0 & 2(k_{1\varepsilon}^2 - k_{0\varepsilon})I \end{bmatrix},$$

satisfying the following Lyapunov equation: $A_\varepsilon P_\varepsilon + P_\varepsilon A_\varepsilon = -Q_\varepsilon$, where P_ε, and Q_ε are positive definite matrices when $k_{1\varepsilon}^2 > k_{0\varepsilon}$.

5.2 Properties of the error equation

In this subsection, we give some useful properties of the perturbations terms $E(q)K_\varepsilon\mathbf{e}$ and Δ of (14). In Lemma 1 we give an important identity and in Lemma 2 we find an upper bound for the non linear term Δ. We will use these Lemmas for proving the boundness of the tracking error.

Lemma 1: The following identities are satisfied:

$$\begin{bmatrix} 0 \\ -E(q)K_\varepsilon\mathbf{e} \end{bmatrix}^T P_\varepsilon \mathbf{e} + \mathbf{e}^T P_\varepsilon \begin{bmatrix} 0 \\ -E(q)K_\varepsilon\mathbf{e} \end{bmatrix} = -2\mathbf{e}^T K_\varepsilon^T E(q) K_\varepsilon \mathbf{e};$$

and

$$\begin{bmatrix} 0 \\ \Delta \end{bmatrix}^T P_\varepsilon \mathbf{e} + \mathbf{e}^T P_\varepsilon \begin{bmatrix} 0 \\ \Delta \end{bmatrix} = 2\mathbf{e}^T K_\varepsilon^T \Delta$$

Proof: (It is trivial.)

Lemma 2: The following inequalities are satisfied

$$\|\Delta_1\| \leq \frac{K_\Phi\|\mathbf{e}\|^2 + K_G\|\mathbf{e}\| + K_0 e^{-\lambda^* t}}{\underline{\mu}}; \quad \|\Delta_2\| \leq \alpha_3 e^{-\lambda^* t},$$

where

$$\alpha_3 = \sqrt{3\alpha_2^2 + 3(\frac{k_0}{\underline{\mu}}\alpha_0)^2 + 3(\frac{k_1}{\underline{\mu}}\alpha_1)^2}.$$
$$K_0 = K_\Phi\alpha_1^2 + K_G\alpha_0;$$

Proof: (We also omit the test.)

It is important to stand out that if all the initial conditions are equals to cero then $\alpha_3 = K_0 = 0$.

We need the following definition [Lewis *et. al.*, 1993].

Definition 1: Let $\mathbf{x}_e$ and $\mathbf{x}_0 = \mathbf{x}(0)$ be the equilibrium state and the initial condition state of the nonlinear system $\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), t)$. We said that $\mathbf{x}_e$ is exponentially and decreasing bounded ($\mathcal{EDB}$) if for each $\delta > 0$, such that $\|\mathbf{x}_e - \mathbf{x}_0\| < \delta$, there exist positives constants T, $K(T,\delta)$ and β such that $\|\mathbf{x}(t) - \mathbf{x}_e\| \leq K(T,\delta)e^{-\beta(t-T)}$, $\forall \ t \geq T$.

Theorem 2 *IF $V(\mathbf{x})$ is a Lyapunov function for any given continuous-time system with the following properties:*

$$\underline{\lambda}\,\|\mathbf{e}(t)\|^2 < V(\mathbf{e}(t)) < \bar{\lambda}\,\|\mathbf{e}(t)\|^2 \tag{15}$$

$$\dot{V}(\mathbf{e}(t)) < -k\,\|\mathbf{e}(t)\|^2 + Z(\mathbf{e}(t)) \tag{16}$$

where $k > 0$, Z is a continuous function of $\mathbf{e}$ which satisfies the following

$$Z(\mathbf{e}(t)) \begin{cases} < 0 \ \text{ if } \ \underline{z}(t) < \|\mathbf{e}(t)\| < \widehat{z} \ \text{ and} \\ = 0 \ \text{ if } \ \mathbf{e}(t)) = \underline{z}(t) \quad \forall \quad t \in [0;\infty), \end{cases} \tag{17}$$

and $\underline{z}$ satisfies

$$\underline{z}(t) \leq z_0 \exp(-\alpha t), \tag{18}$$

for $\alpha > 0$. If

$$\widehat{z} > z_0 \sqrt{\frac{\bar{\lambda}}{\underline{\lambda}}} \quad \text{and} \quad \|\mathbf{e}(0)\| < z_0 \tag{19}$$

then for a value sufficiently large of $T > 0$, we have

$$\|\mathbf{e}(t)\|^2 \leq \bar{z}^2(T) \exp\left(-\widehat{\alpha}(t-T)\right), \forall\ T \leq t < \infty,$$

where

$$\bar{z}(t) \triangleq z_0 \exp(-\alpha t) \sqrt{\frac{\bar{\lambda}}{\underline{\lambda}}} \forall \quad t \in [0; \infty) \tag{20}$$

$$\text{with } \ \alpha = \min\left\{k/\bar{\lambda}, 2\alpha\right\}.$$

This Theorem is based on Theorem 1 of [Lewis *et. al.*, 1993] and it is proved in Appendix .

5.3 Exponentially and Decreasing Boundness of Energy

In this subsection we present sufficient conditions for the parameters $k_{0\varepsilon}$ and $k_{1\varepsilon}$ in order to guarantee the exponentially and decreasing boundness of the error equation (14).

Theorem 3 *The closed loop system (14) is exponentially and decreasing bounded for fixed γ s.t. $0 < \gamma < 1$, if*

$$k_{1\varepsilon}^2 > k_{0\varepsilon} \tag{21}$$

$$\gamma \lambda_m \{Q_\varepsilon\} > \frac{2\kappa_\varepsilon K_G}{\underline{\mu}} + 2\kappa_\varepsilon \sqrt{\frac{K_\Phi \alpha_4}{\underline{\mu}}} \tag{22}$$

$$\sqrt{\frac{\lambda_m\{P_\varepsilon\}}{\lambda_M\{P_\varepsilon\}} \frac{\underline{\mu}\alpha_4}{K_\Phi}} < \frac{K_S + \sqrt{K_S^2 - \frac{4K_\Phi \kappa_\varepsilon^2 \alpha_4}{\underline{\mu}}}}{\frac{2K_\Phi \kappa_\varepsilon}{\underline{\mu}}} \tag{23}$$

with $K_S = \gamma \lambda_m \{Q_\varepsilon\} - \frac{2\kappa_\varepsilon K_G}{\underline{\mu}}$, then for some $T > 0$ we have

$$\|\mathbf{e}(t)\|^2 \leq \frac{\lambda_m\{P_\varepsilon\}}{\lambda_M\{P_\varepsilon\}} \frac{\underline{\mu}\alpha_4 e^{-\lambda^* T}}{K_\Phi} e^{-\upsilon(t-T)} \quad \forall \quad T \leq t < \infty$$

where

$$\alpha_4 = \sqrt{\frac{K_\Phi^2 \|\mathbf{e}(0)\|^2}{\underline{\mu}^2} + 2(\alpha_3^2 + (\frac{K_0}{\underline{\mu}})^2)}, \quad \upsilon = \min\{\frac{(\gamma-1)\lambda_m\{Q_\varepsilon\}}{\lambda_M\{P_\varepsilon\}}, \lambda^*\}. \tag{24}$$

Proof: This Theorem is proved in four steps.
First Step: We propose the following Lyapunov equation[3]

$$V(\mathbf{e}) = \frac{1}{2}\mathbf{e}^T P_\varepsilon \mathbf{e}, \tag{25}$$

from **Remark 3**, we have that V satisfies:

$$\frac{\lambda_m\{P_\varepsilon\}}{2} \|\mathbf{e}\|^2 < V(\mathbf{e}) < \frac{\lambda_M\{P_\varepsilon\}}{2} \|\mathbf{e}\|^2 \ . \tag{26}$$

Second Step: Let us compute the derivative of V. Using Lemma 1, we have:

$$\dot{V}(\mathbf{e}) = -\frac{\mathbf{e}^T Q_\varepsilon \mathbf{e} + 2\mathbf{e}^T K_\varepsilon^T E(q) K_\varepsilon \mathbf{e}}{2} + \mathbf{e}^T K_\varepsilon^T \Delta. \tag{27}$$

Now we proceed to find an upper bound of $\dot{V}$. To do this, we need to take into the account Lemmas 1 and 2, property **P.6** and definition of α_4 given in (24)**.** This yields after some manipulations the following inequality:

$$\dot{V}(\mathbf{e}) \leq -\frac{\lambda_m\{Q_\varepsilon\}}{2} \|\mathbf{e}\|^2 + \kappa_\varepsilon \|\mathbf{e}\| \left(\frac{K_\Phi \|\mathbf{e}\|^2 + K_G \|\mathbf{e}\|}{\underline{\mu}} + \alpha_4 e^{-\lambda^* t}\right);$$

with $\kappa_\varepsilon = \max\{k_{0\varepsilon}, k_{1\varepsilon}\}$.

In order to use Theorem 2, let´s separate $\dot{V}$ in two terms:

$$\dot{V}(\mathbf{e}) \leq \dot{V}_\gamma(\mathbf{e}) + \dot{V}_\varepsilon(\mathbf{e}) \tag{28}$$

where

$$\dot{V}_\gamma(\mathbf{e}) = -\frac{(\gamma-1)\lambda_m\{Q_\varepsilon\}}{2} \|\mathbf{e}\|^2 \tag{29}$$

$$\dot{V}_\varepsilon(\mathbf{e}) = \frac{K_\Phi \kappa_\varepsilon \|\mathbf{e}\|^3}{\underline{\mu}} + \left(\frac{\kappa_\varepsilon K_G}{\underline{\mu}} - \frac{\gamma \lambda_m\{Q_\varepsilon\}}{2}\right) \|\mathbf{e}\|^2 + \kappa_\varepsilon \alpha_4 e^{-\lambda^* t} \|\mathbf{e}\| . \tag{30}$$

Third Step: We need Lemma 1 of [Lewis *et. al.*, 1993], recalled hereafter:
Lemma 3: For some arbitrary z, if

$$\beta_3 > \zeta_1 + \sqrt{4\zeta_0\zeta_2}$$

then

$$\zeta_2 z + (\zeta_1 - \beta_3) z + \zeta_0 < 0 \quad \text{for} \quad z_1 < z < z_2,$$

where $\zeta_0, \zeta_1, \zeta_2$ and β_3 are positives constants

$$z_1 = \frac{(\beta_3 - \zeta_1) - \sqrt{(\beta_3 - \zeta_1)^2 - 4\zeta_0\zeta_2}}{2\zeta_2}; \quad z_2 = \frac{(\beta_3 - \zeta_1) + \sqrt{(\beta_3 - \zeta_1)^2 - 4\zeta_0\zeta_2}}{2\zeta_2}.$$

.

In view of (22) we have:

$$\gamma \lambda_m \{Q_\varepsilon\} > \frac{2\kappa_\varepsilon K_G}{\underline{\mu}} + 2\kappa_\varepsilon \sqrt{\frac{K_\Phi \alpha_4 e^{-\lambda^* t}}{\underline{\mu}}} \tag{31}$$

[3] To simplify the notation, we write $\mathbf{e}$ instead of $\mathbf{e}(t)$.

then from the above Lemma we get:

$$\frac{K_\Phi \kappa_\varepsilon \|\mathbf{e}\|^2}{\underline{\mu}} + \left(\frac{\kappa_\varepsilon K_G}{\underline{\mu}} - \frac{\gamma\lambda_m\{Q_\varepsilon\}}{2}\right)\|\mathbf{e}\| + \alpha_4 \kappa_\varepsilon e^{-\lambda^* t} < 0, \tag{32}$$

for $e_1(t) < \|\mathbf{e}\| < \bar{e}$. Namely

$$\dot{V}_\varepsilon(\mathbf{e}) < 0 \quad \text{for} \quad e_1 < \|\mathbf{e}\| < \bar{e} \tag{33}$$

where$K_S = \gamma\lambda_m\{Q_\varepsilon\} - \frac{2\kappa_\varepsilon K_G}{\underline{\mu}}$

$$\bar{e} = \frac{K_S + \sqrt{K_S^2 - \frac{4K_\Phi \kappa_\varepsilon^2 \alpha_4}{\underline{\mu}}}}{\frac{2K_\Phi \kappa_\varepsilon}{\underline{\mu}}} \tag{34}$$

$$e_1 = \frac{K_S - \sqrt{K_S^2 - \frac{4K_\Phi \kappa_\varepsilon^2 \alpha_4 e^{-\lambda^* t}}{\underline{\mu}}}}{\frac{2K_\Phi \kappa_\varepsilon}{\underline{\mu}}} \tag{35}$$

Fourth Step: Let us first note that

$$\dot{V}_\varepsilon(\mathbf{e}) \begin{cases} < 0 & \text{if} \quad \sqrt{\frac{\underline{\mu}\alpha_4}{K_\Phi}} e^{-\frac{\lambda^*}{2}t} < \|\mathbf{e}\| < \bar{e} \\ \forall & t \in [0;\infty), \\ = 0 & \text{if} \quad \|\mathbf{e}\| = e_1(t) \quad \forall \quad t \in [0;\infty). \end{cases} \tag{36}$$

Indeed, considering Pythagorean Theorem in (35) ($A \leq \sqrt{A^2 - 4C} + 2\sqrt{C}$) we get*:*

$$e_1 \leq \sqrt{\frac{\underline{\mu}\alpha_4}{K_\Phi}} e^{-\frac{\lambda^*}{2}t} \ . \tag{37}$$

and thus, from (33) and (37), we have (36).

Let´s mention that the Assumptions of Theorem 3 are held for the Lyapunov function V. Indeed:

A) (15) follows from (26). B) (16) follows from (28),(29) and (30). C) (17) follows from (36). D) (18) follows from (37).

Now, from inequality (23) and definition of α_4 (see 24), we have that (compare with (19)):

$$\|\mathbf{e}(0)\| < \sqrt{\frac{\underline{\mu}\alpha_4}{K_\Phi}} \quad \text{and} \quad \sqrt{\frac{\lambda_m\{P_\varepsilon\}}{\lambda_M\{P_\varepsilon\}} \frac{\underline{\mu}\alpha_4}{K_\Phi}} < \bar{e}.$$

Then, Theorem 2 implies :

$$\|\mathbf{e}\|^2 \leq \frac{\lambda_m\{P_\varepsilon\}}{\lambda_M\{P_\varepsilon\}} \frac{\underline{\mu}\alpha_4 e^{-\lambda^* T}}{K_\Phi} e^{-\upsilon(t-T)} \quad \forall \quad t \in [T,\infty) \, .$$

.

5.4 Simulations Results

To evaluate the performance of the proposed control law (4), we carried out some digital computer simulations for different values of the parameter ε. We have set the original system and controller parameters to be,

$$M_p = 5 \text{ kg}; M_v = 1 \text{ kg}; r = 1 \text{ m}; L = 2\text{m}, \\ k_0 = 1,\ k_1 = 1 \text{ and } \ \beta = 2.$$

With the following initial conditions: $x(0) = -0.2$m, $y(0) = 1.1$, $\dot{x}(0) = 10$m/s and $\dot{y}(0) = 10$m/s, and being the reference: $r_x = 0.5$m and $r_y = 1.8$m. We write $q^T = [x \quad y]$, $\zeta^T = [\zeta_x \quad \zeta_y]$ and $f_q^T = [f_x \quad f_y]$.

In figure 2 ,we show the tracking error $x(t) - \zeta_x(t)$ and $x(t) - \zeta_y(t)$ for three different values of the parameter ε. The highest, medium and lowest overshoot corresponding to $\varepsilon = 5$, $\varepsilon = 1$ and $\varepsilon = 0.1$, respectively.

In figure 3, we show the behavior of q(m) and ζ^T(m) for $\varepsilon = 0.1$. And in figure 4 we show f_q^T(N).

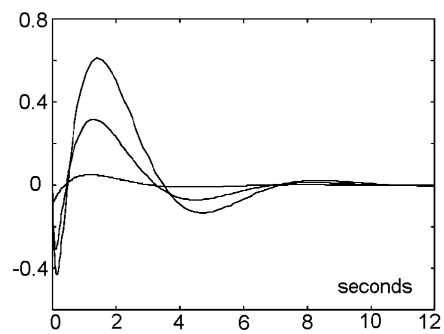

Fig. 2a $x - \zeta_x;\ \varepsilon = 5, 1, 0.1$

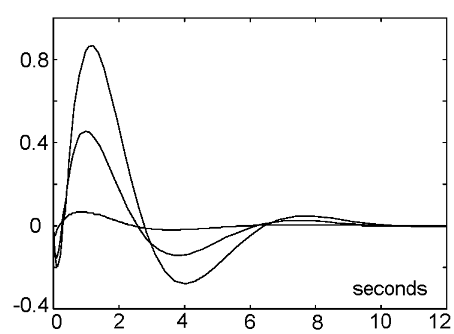

Fig. 2b $y - \zeta_y; \varepsilon = 5, 1, 0.1$

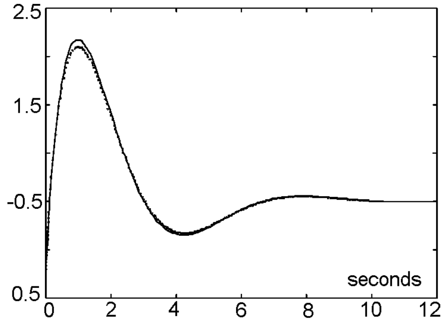

Fig. 3a x, ζ_x; $\varepsilon = 0.1$;

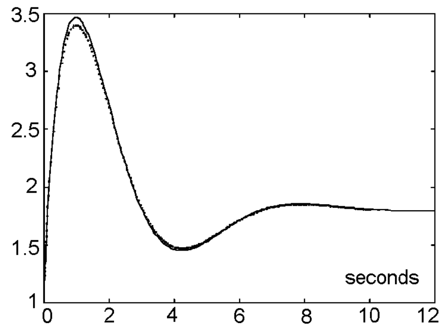

Fig. 3b y, ζ_y; $\varepsilon = 0.1$;

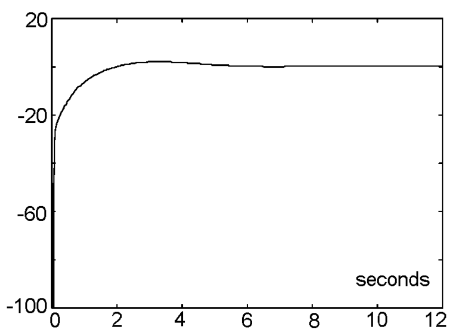

Fig. 4a f_x; $\varepsilon = 0.1$;

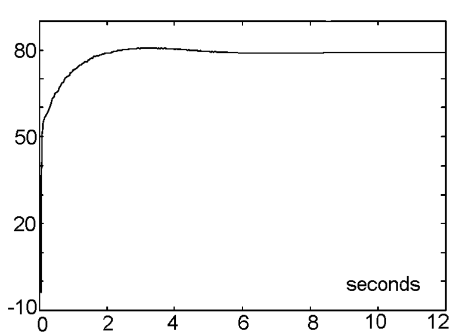

Fig. 4b f_y; $\varepsilon = 0.1$;

6 Concluding Remarks

In this paper we have proposed a linear stabilizable implicit control law for a two degree of freedom parallel manipulator. This controller needs to adjust four parameters, namely ε, β, k_0, and k_1. The constant ε determines the stability (see Theorem 1) , β determines the smoothness of the control action (2) and the constants k_0 and k_1 determine the dynamics of the implicit action Note that Theorem 1 states that the system is $\mathcal{AS}$ when ε is less than a bound which is directly related with the Lipschitz constant K_G; namely the $\mathcal{AS}$ is directly related with the weight linear density of the action load.

Theorem 3 states that, when the initial conditions are close to the equilibrium point, the tracking error, $\mathbf{e}(.)$, between the states $q(.)$ of the closed-loop system (7) and the desired trajectory $\zeta(.)$ (see 10) is bounded by an exponential decreasing function. Another way to see Theorem 3 is that given any finite initial condition and some fix parameters k_0,k_1 and β , we can find a parameter ε such that the error is bounded by an exponential decreasing function as ε tends to zero. The proofs of asymptotic stabilty and the exponential model macthing were carried on by means of the well known Lyupanov´s second method

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7 Appendix

Proof of the Theorem 2: Let us first separate the space of $\mathbf{e}$ in two regions using the negativeness of the function $Z(.)$, as follows

$$\begin{aligned} S_e &= \{\mathbf{e}(t) \mid \; z_0 e^{-\alpha t} \leq \|\mathbf{e}(t)\| \leq \widehat{z}\}; \\ S_i &= \{\mathbf{e}(t) \mid \|\mathbf{e}(t)\| < \; z_0 e^{-\alpha t}\}. \end{aligned} \tag{38}$$

We proceed to consider two interesting cases:

First Case: Let us first consider that $\mathbf{e}(t)$ never leaves S_i, then

$$\|\mathbf{e}(t)\| \leq z_0 e^{-\alpha t} < \bar{z}(t) \quad \forall \quad t \in [0; \infty).$$

Second Case: Let us next consider that $\mathbf{e}(t)$ comes into S_e for some T. Then by continuity of $\mathbf{e}$ we have

$$\|\mathbf{e}(T)\| = z_0 e^{-\alpha T} \quad ,\dot{V}(\mathbf{e}(t)) < -k \|\mathbf{e}(t)\|^2 \quad \forall \; t \geq t + \Delta t. \tag{39}$$

Integrating the last inequality for $\Delta t > 0$, and using properties (15),(16) of V we get (recall a definition of $\bar{z}(.)$ (18))

$$\|\mathbf{e}(T+\Delta t)\|^2 < \bar{z}^2(T)\exp\left(\frac{-k}{\bar{\lambda}}\Delta t\right), \qquad (40)$$

using now conditions (19) in the above inequality, we have that

$$\|\mathbf{e}(T+\Delta t)\|^2 < \bar{z}^2(T)\exp\left(\frac{-k}{\bar{\lambda}}\Delta t\right) < \hat{z}^2$$

Then $\mathbf{e}(T+\Delta t)$ never lives S_e for any arbitrary $\Delta t > 0$.

Let us finally analyze inequality (40). For this we need to consider the following two cases

$$\bar{z}^2(T+\Delta t) < \bar{z}^2(T)\exp(\frac{-k_v}{\bar{\lambda}}\Delta t); \quad \text{if} \quad k/\underline{\lambda} < 2\alpha,$$
$$\bar{z}^2(T)\exp(\frac{-k_v}{\bar{\lambda}}\Delta t) \leq \bar{z}^2(T+\Delta t) \quad \text{otherwise.}$$

Then by induction:

$$\|\mathbf{e}(t)\|^2 \leq \bar{z}^2(T)\exp[-\hat{\alpha}(t-T)] \ \ \forall \ \ T \leq t < \infty,$$

where $\hat{\alpha} = \min\{\frac{k}{\underline{\lambda}}, 2\alpha\}$.

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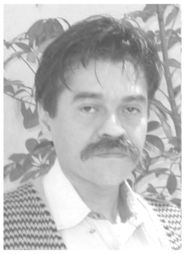

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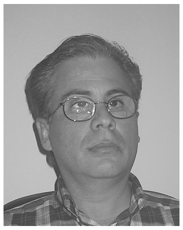

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